Analysis and Intrinsic Properties of the General Nonuniform Transmission Line

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Abstract—**A new formulation for the analysis of the general nonuniform transmission line is presented. A characterization of the line in terms of its *ABCD* parameters is given. Explicit formulas for these parameters are obtained from which any network function can be calculated to any degree of accuracy. The new formulation leads to a description of the line in terms of a Sturm-Liouville equation from which all of the natural frequencies of the line can be found and their distribution bounds and asymptotic behavior specified. The *ABCD* parameters are shown to be entire functions of order 1 and genus 1.**

I. Introduction

NONUNIFORM transmission lines have been used extensively as impedance transformers and more recently as resonators [1], [2] filters [3], and delay equalizers [4]. Except for a few special cases, and for cases in which the lines are "smooth," no closed form solution exists. The purpose of this paper is to present a new formulation from which a general method of analysis and the intrinsic properties of an arbitrarily tapered transmission line are deduced. In particular, a characterization of the line in terms of its *ABCD* parameters is offered. Explicit formulas of these parameters are obtained from which any network function can be evaluated to any degree of accuracy. The new formulation leads also to a description of the line in terms of a Strum-Liouville equation from which all of the natural frequencies of the line can be found and their distribution, bounds, and asymptotic behavior specified. Finally, the analytic properties of the *ABCD* parameters are derived, and it is shown that the parameters are all entire functions of the complex frequencies of order 1 and genus 1.

There exists a large amount of excellent literature on nonuniform transmission lines [23]–[25] and propagation through nonuniform layers [26], [27]. Some of these papers tackle the analysis problem for a given transmission line through integral equation formulations without extracting the properties of the network in general. Our purpose here is to present an analysis that would enable us to extract some intrinsic properties of the general nonuniform lossless transmission line.

There exists also a large number of papers on the synthesis of nonuniform transmission lines [23], [28]–[34]. The authors have also considered the synthesis problem in another paper [20] where they present realizability conditions in the frequency domain.

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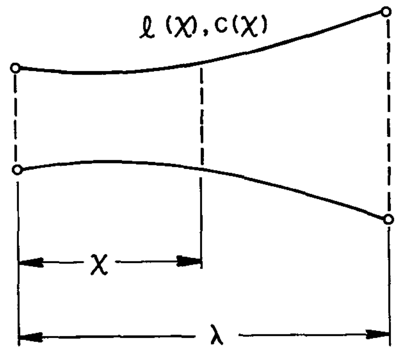

Fig. 1. A section of a nonuniform lossless transmission line.

II. Characterization—Explicit Formulas for *ABCD* Parameters

Consider a section of nonuniform lossless transmission line (Fig. 1) having a series inductance per unit length $l(x)$ and a shunt capacitance per unit length $c(x)$, $0 \leq x \leq \lambda$, where λ is the length of the line. Except for the obvious restriction that $l(x)$ and $c(x)$ be non-negative everywhere, no other restrictions are imposed on $l(x)$ and $c(x)$. Indeed, both may have impulses so that lumped elements as well as distributed elements are allowed along the line.

We choose to study the terminal behavior of the line in terms of its *ABCD* parameters. Let the length λ be divided into n intervals of equal length $\Delta x = \lambda/n$. The *ABCD* parameters of the kth elementary section are

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = \begin{bmatrix} 1 & s\Delta x \cdot l(k\Delta x) \\ s\Delta x \cdot c(k\Delta x) & 1 \end{bmatrix}.$$

The overall *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \lim_{n\to\infty} \prod_{k=1}^{n} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}.$$

Expanding the product and replacing sums by integrals, we get the explicit formulas

$$A = 1 + \sum_{n=1}^{\infty} a_n s^{2n} \tag{1}$$

$$B = s\int_0^{\lambda} l(x)dx + s\sum_{n=1}^{\infty} b_n s^{2n} \tag{2}$$

$$C = s\int_0^{\lambda} c(x)dx + s\sum_{n=1}^{\infty} c_n s^{2n} \tag{3}$$

$$D = 1 + \sum_{n=1}^{\infty} d_n s^{2n} \tag{4}$$

$$a_n = \int_0^\lambda c(x_{2n}) \int_0^{x_{2n}} l(x_{2n-1}) \int_0^{x_{2n-1}} c(x_{2n-2}) \cdots \cdots \int_0^{x_3} c(x_2) \int_0^{x_2} l(x_1) dx_1 dx_2 \cdots dx_{2n} \tag{5}$$

$$b_n = \int_0^\lambda l(x_{2n+1}) \int_0^{x_{2n+1}} c(x_{2n}) \int_0^{x_{2n}} l(x_{2n-1}) \cdots \cdots \int_0^{x_3} c(x_2) \int_0^{x_2} l(x_1) dx_1 dx_2 \cdots dx_{2n+1} \tag{6}$$

$$c_n = \int_0^\lambda c(x_{2n+1}) \int_0^{x_{2a+1}} l(x_{2n}) \int_0^{x_{2n}} c(x_{2n-1}) \cdots \cdots \int_0^{x_3} l(x_2) \int_0^{x_2} c(x_1) dx_1 dx_2 \cdots dx_{2n+1} \tag{7}$$

$$d_n = \int_0^\lambda l(x_{2n}) \int_0^{x_{2n}} c(x_{2n-1}) \int_0^{x_{2n-1}} l(x_{2n-2}) \cdots \cdots \int_0^{x_3} l(x_2) \int_0^{x_2} c(x_1) dx_1 dx_2 \cdots dx_{2n}. \tag{8}$$

In a physical line, $l(x)$ and $c(x)$ are not identically zero and $l(x) \geq 0$ and $c(x) \geq 0$, so that

$$a_n > 0, b_n > 0, c_n > 0, \quad \text{and} \quad d_n > 0.$$

Moreover, let $l(x)$ and $c(x)$ be bounded so that

$$l(x) < L_0 \quad \text{and} \quad c(x) < C_0.$$

We have, by integration,

$$a_n < \frac{(L_0 C_0 \lambda^2)^n}{(2n)!}$$

$$b_n < \frac{(L_0 C_0 \lambda^2)^n L_0 \lambda}{(2n+1)!}$$

$$c_n < \frac{(L_0 C_0 \lambda^2)^n C_0 \lambda}{(2_n+1)!}$$

$$d_n < \frac{(L_0 C_0 \lambda^2)^n}{(2n)!}.$$

Therefore, the four series (1), (2), (3), and (4) are all uniformly convergent and the *ABCD* parameters are entire functions of *s*.

Having the explicit formulas of the *ABCD* parameters, we can now evaluate any network function of the line, terminated or unterminated. For example, the input impedance is $(AZ_L+B)/(CZ_L+D)$, where Z_L is the terminating impedance. The input reflection coefficient is $[(RC-A)Z_L+(RD-B)]/[(RC+A)Z_L+(RD+B)]$, where R is the source impedance. Note that in contrast with the previous formulation where the reflection coefficient and the input impedance must be obtained from the solution of nonlinear differential equations [5], [6] the present formulation leads to explicit expressions which can be evaluated to any desired degree of accuracy. Moreover, $l(x)$ and $c(x)$ need not be continuous functions of x. It should also be noticed that the series (1), (2), (3), and (4) for A, B, C, and D are alternating for $s=j\omega$ and they are quite rapidly convergent for frequencies small compared to the smallest natural frequency of the line short circuited at both ends. For large frequencies the asymptotic formulas developed in later sections of this paper should be used. For intermediate frequencies the series (1), (2), (3), and (4) may not be as rapidly convergent as the iterative procedure presented in the literature [23]–[25] based on a Volterra integral equation formulation. Yet, in contrast with the latter, it gives explicit formulas and it should mainly be viewed as a method which enables us to extract general analytical properties of the nonuniform lines such as the ones presented in the later sections of this paper. The present method is also useful as a computational means since it can very easily be incorporated in a computer program. Note also that the coefficients $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ may be used for the computation of the natural frequencies of a section of nonuniform line with an LC-impedance termination as described in Section 7. The derivation of the series for $A(s)$, $B(s)$, $C(s)$, and $D(s)$ as outlined here is extremely simple. The step-by-step method of Schelkunoff [19] would yield identical reesults. Similar results for RC nonuniform lines have been presented by the authors [35], and by Hellstrom [22] independently.

III. The Sturm-Liouville Equation

To gain a deeper insight into the properties of the general nonuniform line, we shall transform the usual telegraphists' equations into a Sturm-Liouville equation. To this end, we make an important change of variables. We let

$$L(x) = \int_0^x l(y) dy \tag{9}$$

$$C(x) = \int_0^x c(y) dy. \tag{10}$$

$L(x)$ is the cumulative inductance along the line, and $C(x)$ is the cumulative capacitance. Let the total inductance of the line be L_T and the total capacitance be C_T. Clearly, we have $L(\lambda) = L_T$ and $C(\lambda) = C_T$.

It is convenient to express C as a function of L. We define a function $\sigma(L)$:

$$\begin{aligned} C &= \sigma(L) = \sigma(L-0), \qquad \text{for } 0 < L < L_T \\ \sigma(0) &= 0 \\ \sigma(L_T) &= C_T \end{aligned} \tag{11}$$

so that $\sigma(L)$ implicitly specifies the nonuniformity of the line. Note that

$$\sigma'(L) = d\sigma/dL = c/l = 1/K^2 \tag{12}$$

where $K(L)$ is the "local" characteristic impedance along the line. $\sigma(L)$ is a nondecreasing function of L and, therefore, it always has a unique inverse.

$$L = \tau(C), \quad C \in (0, C_T) \tag{13}$$

$$\tau'(C) = d\tau/dC = l/c = K^2. \tag{14}$$

Note that lumped shunt capacitors correspond to jumps of $\sigma(L)$ and lumped series inductors correspond to jumps of $\tau(C)$ (Fig. 2). For an LC ladder network, $\sigma(L)$ is piecewise constant (Fig. 3). Using $\sigma(L)$, we now have a unified description of general LC lumped ladder networks or distributed networks or mixed structures. The spatial variable has been suppressed.

In terms of L and C, the usual telegraphists' equations on the voltage V and current I along the line are transformed into Sturm-Liouville equations

$$\frac{d^2V}{dL^2} - s^2\sigma'(L)V = 0 \qquad (15)$$

$$\frac{d^2I}{dC^2} - s^2\tau'(C)I = 0. \qquad (16)$$

Note that the equations are characterized by σ' and τ' so that nonuniform lines with the same $C=\sigma(L)$, $L\in(0, L_T)$, have identical electrical characteristics at the terminals, i.e., they are equivalent. For example, the following two nonuniform lines have the same terminal behavior

1) $l(x) = e^x$, $\quad c(x) = 2e^{-x}$, $\quad \lambda = ln2$

2) $l(x) = 2(x+1)$, $\quad c(x) = \dfrac{4}{(x+1)^3}$, $\quad \lambda = \sqrt{2} - 1$.

IV. Natural Frequencies

Walker and Wax obtained an iterative scheme to compute the natural frequencies of a tapered line with a reactive termination [5]. Since any reactance can be represented by an LC ladder of the Cauer form, the reactive termination can be incorporated into the function $\sigma(L)$ or $\tau(C)$ in the description of the nonuniform line and we may henceforth regard all lines as unterminated.

We shall presently show that the natural frequencies of a nonuniform line are the eigenvalues of a Sturm-Liouville problem with zero boundary conditions and are in fact related to the zeros of the *ABCD* parameters.

The zeros and poles of an impedance function are the short-circuit and open-circuit natural frequencies, respectively, of the network at the pertinent driving point. Since $A(s)/C(s)$ = input impedance with output open-circuited and $B(s)/D(s)$ = input impedance with output short-circuited, the zeros of the parameters $A(s)$, $B(s)$, $C(s)$, and $D(s)$ are the natural frequencies of the networks of Fig. 4(a), (b), (c), and (d), respectively.

Denote by $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$, $n=1, 2, \cdots$, respectively, the real frequencies for which the $A(j\omega)$, $B(j\omega)$, $C(j\omega)$, and $D(j\omega)$ vanish. So that, 1) $\{\alpha_n\}$, $n=1, 2, \cdots$, are the natural frequencies of the section of the lossless transmission line short-circuited at the sending end and open-circuited at the receiving end, [Fig. 4(a)], 2) $\{\beta_n\}$, $n=1, 2, \cdots$, are the natural frequencies of the line short-circuited at both ends [Fig. 4(b)], 3) $\{\gamma_n\}$ are the natural frequencies of the line open-circuited at both ends [Fig. 4(c)], and 4) δ_n are the natural frequencies of the line which is open-circuited at the receiving end [Fig. 4(d)]. In the following we will be referring to the natural frequencies of the different configurations of the line as zeros of one of the *ABC* or *D* parameters.

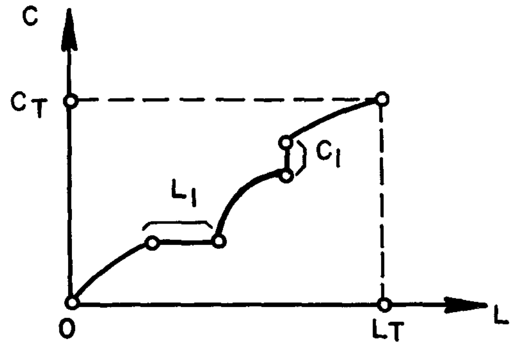

Fig. 2. The function $C=\sigma(L)$.

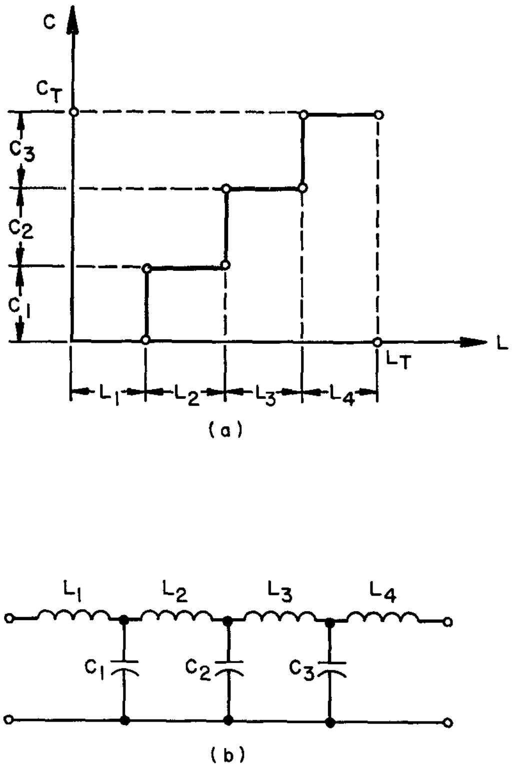

Fig. 3. The function $C=\sigma(L)$ for the ladder network (b).

With respect to Fig. 4(b) and from (15), the zeros of $B(s)$ are the eigenvalues of the following Sturm-Liouville problem

$$\begin{cases} \dfrac{d^2V}{dL^2} + \mu^2\sigma'(L)V = 0 \\ V(0) = V(L_T) = 0. \end{cases}$$

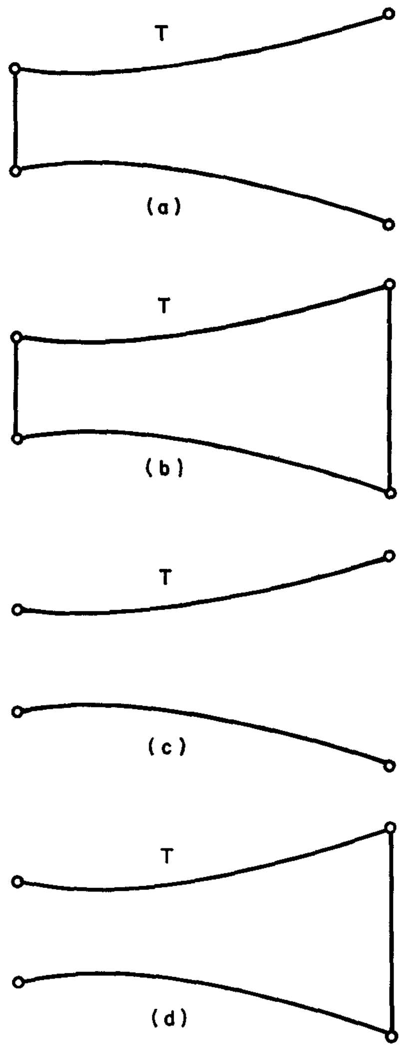

Fig. 4. The boundary conditions.

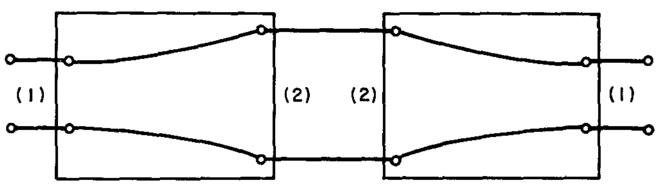

Fig. 5. A back-to-back arrangement of a section of the line.

Similarly, the zeros of $C(s)$ are the eigenvalues of the following Sturm-Liouville problem

$$\begin{cases} \dfrac{d^2I}{dC^2} + \mu^2\tau'(c)I = 0 \\ I(0) = I(C_T) = 0. \end{cases}$$

As to the zeros of $A(s)$ and $D(s)$, we consider Fig. 5 which is a back-to-back arrangement of identical halves coinciding with the given section of the nonuniform LC transmission line of Fig. 4.

Let A', B', C', and D' be the chain parameters of the two-port in Fig. 5. Then

$$B'(s) = 2A(s)B(s) \tag{17}$$

$$C'(s) = 2C(s)D(s). \tag{18}$$

It is known that the zeros of $A(s)$ and $B(s)$, as well as the zeros of $C(s)$ and $D(s)$, interlace, so that

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots \tag{19}$$

and

$$0 < \delta_1 < \gamma_1 < \delta_2 < \gamma_2 < \cdots \tag{20}$$

Consequently, from relations (17), (18), (19), and (20) it follows that the zeros of $A(s)$ are the *odd*-order zeros of $B'(s)$, and the zeros of $D(s)$ are the *odd*-order zeros of $C'(s)$.

Thus, the zeros of $A(s)$ are the *odd*-order eigenvalues of the following Sturm-Liouville boundary value problem with zero boundary conditions at both ends:

$$\begin{gathered} \frac{d^2V}{dL^2} + \mu^2\sigma'(L)V = 0 \\ 0 \le L \le 2L_T \\ V(0) = V(2L_T) = 0 \end{gathered} \tag{21}$$

where for $L_T < L \le 2L_T$, $\sigma'(L) = \sigma'(2L_T - L)$.

Whereas the zeros of $D(s)$ are the *odd*-order eigenvalues of the following Sturm-Liouville system:

$$\begin{gathered} \frac{d^2I}{dC^2} + \mu^2\tau'(C)I = 0 \\ 0 \le C \le 2C_T \\ I(0) = I(2C_T) = 0 \end{gathered} \tag{22}$$

where for $C_T < C \le 2C_T$, $\tau'(C) = \tau'(2C_T - C)$.

V. Bounds on the Natural Frequencies

In the previous section, the zeros of the *ABCD* parameters were identified with the eigenvalues of boundary value problems of the form

$$y'' + \mu^2\rho(x)y = 0, \quad y(0) = y(a) = 0. \tag{23}$$

The mathematical literature on eigenvalue problems is voluminous [7]–[18]. We shall make liberal use of many results which are pertinent to the problem at hand.

5.1: Arbitrary nonuniform lossless transmission line with specified total inductance L_T and total capacitance C_T.

In (23) denote $\int_0^a \rho(x)dx = M$. It is well known [7], [14]–[18] that the smallest eigenvalue $\mu_1[\rho]$ of the boundary value problem (23) satisfies the inequality

$$\mu_1[\rho] \ge \frac{2}{\sqrt{aM}} \cdot \tag{24}$$

For the problem at hand and with respect to the first zero of $B(s)$ compare the differential equation (15) with (23). We have: $x \leftrightarrow L$, $\rho(x) \leftrightarrow \sigma'(L)$, $a \leftrightarrow L_T$ and $M \leftrightarrow \int_0^{L_T} \sigma'(L) dL = C_T$. Therefore, applying (3.20) we have:

$$\beta_1 \geq \frac{2}{\sqrt{L_T C_T}} \cdot \tag{25}$$

This inequality is the best possible, i.e., there exist tapers for which the value of β_1 is arbitrarily close to $2/\sqrt{L_T C_T}$. In fact, the equality is attained for the T-network in Fig. 6.

It is known [7], [14]–[18] that the higher-order eigenvalues of the boundary value problem (23) satisfy the inequality

$$\mu_n[\rho] \geq \frac{2n}{\sqrt{aM}} \cdot \tag{26}$$

Comparing (15), (16), (21), and (22) with (23) we easily get

$$\alpha_n, \delta_n \geq \frac{2n-1}{\sqrt{L_T C_T}} \tag{27}$$

$$\beta_n, \gamma_n \geq \frac{2n}{\sqrt{L_T C_T}} \tag{28}$$

$$n = 1, 2, \cdots$$

Equalities are attained for α_n, β_n, γ_n, and δ_n for the ladder networks in Fig. 7(a), (b), (c), and (d), respectively.

5.2: As a second case, suppose that the only thing specified about the taper is $\Delta = \int_0^\lambda \sqrt{lc}\, dx \leq L_T C_T$. Without any additional information about the taper it is not difficult to construct examples showing that the product $\beta_1 \Delta$ can get arbitrarily small. However, the additional requirement that c/l is a non-negative monotonic function of $L \in L^1[0, L_T]$ is sufficient to produce a lower bound. In fact using the results of Nehari [17] about the eigenvalues of the boundary value problem (23) in the Sturm-Liouville equations (15) and (16), we get, respectively,

$$\beta_1 \geq \frac{\pi}{2\Delta} \quad \text{and} \quad \gamma_1 \geq \frac{\pi}{2\Delta} \cdot \tag{29}$$

Note that these inequalities are the best possible with the present constraints [17]. In general, if $\sigma'(L) \in L^1[0, L_T]$ and if the interval $[0, L_T]$ can be decomposed into $k+1$ on which $\sigma'(L) = c/l$ is a monotonic function of L, the zeros obey the following inequalities

$$\alpha_n, \delta_n > (n - k - 1)\pi/2\Delta; \; \beta_n, \gamma_n > (n - k)\pi/2\Delta \quad \text{for } n > k. \tag{30}$$

5.3: Let $L_T > 0$, $C_T > 0$, $K_M > 0$, $K_m > 0$ be numbers subject to the condition

$$K_m^2 \leq L_T/C_T \leq K_M^2.$$

Consider the class of LC nonuniform transmission lines with total series inductance L_T, total shunt capacitance C_T, and local characteristic impedance $K(x) = \sqrt{l(x)/c(x)}$ such that

$$K_m \leq K(x) \leq K_M$$

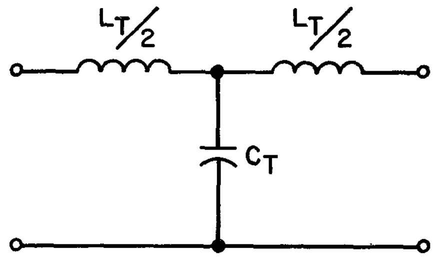

Fig. 6. The ladder for which β_1 is minimum.

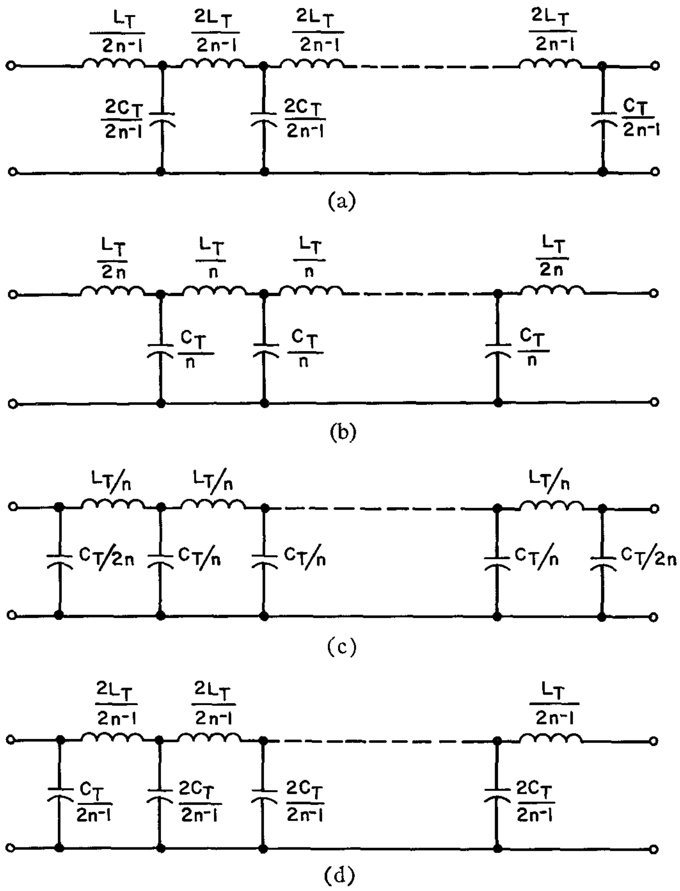

Fig. 7. The ladders for which the natural frequencies are minimized.

for $0 \leq x \leq \lambda$. Denote this class of nonuniform lossless lines by $T(L_T, C_T, K_M, K_m)$. Since $\sigma'(L) = c/l = 1/K^2(L)$ we have

$$\frac{1}{K_M^2} \leq \sigma'(L) \leq \frac{1}{K_m^2},$$

i.e., all the points of the plot of the function $C = \sigma(L)$ must lie inside the parallelogram whose sides have slopes $1/K_m^2$ and $1/K_M^2$ and two of its vertices coincide with the origin and the point (L_T, C_T), respectively (Fig. 8).

Consider the zeros of the $ABCD$ parameters as functionals of $\sigma(L)$. The problem is to find for what functions $\sigma(L) \in T$ (L_T, C_T, K_M, K_m) these zeros attain extreme values and to

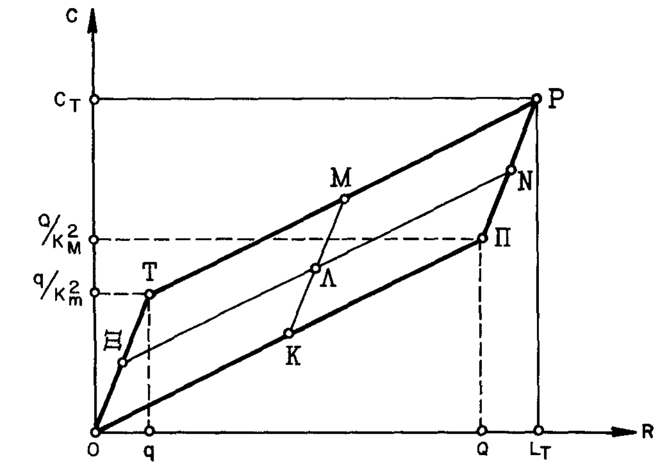

Fig. 8. Diagram for the class $T\,(L_T, C_T, K_M, K_m)$.

determine the corresponding upper and lower bounds if they exist. We can give the answers to this question using Krein's results [16] for the eigenvalues of the boundary value problem (23) where $h \leq \rho(x) \leq H$.

The following theorem states the results.

Theorem 1:

For an LC transmission line $\in T(L_T, C_T, K_M, K_m)$ the maximum and minimum values of the *ABCD* parameters are given by the following relations:

a)
$$\max \alpha_n(\sigma) = \max \delta_n(\sigma) = \frac{(2n-1)K_M}{Q}\,\phi\left(\frac{K_M}{K_m}, \frac{q}{Q}\right) \tag{31}$$

$$\min \alpha_n(\sigma) = \min \delta_n(\sigma) = \frac{(2n-1)K_m}{q}\,\phi\left(\frac{K_m}{K_M}, \frac{Q}{q}\right) \tag{32}$$

b)
$$\max \beta_n(\sigma) = \max \gamma_n(\sigma) = \frac{2nK_M}{Q}\,\phi\left(\frac{K_M}{K_m}, \frac{q}{Q}\right) \tag{33}$$

$$\min \beta_n(\sigma) = \min \gamma_n(\sigma) = \frac{2nK_m}{q}\,\phi\left(\frac{K_m}{K_M}, \frac{Q}{q}\right) \tag{34}$$

where, with reference to Fig. 8,

$$q = K_m^2\frac{K_M^2C_T - L_T}{K_M^2 - K_m^2}, \quad Q = K_M^2\frac{L_T - K_m^2C_T}{K_M^2 - K_m^2} \tag{35}$$

and $\phi(p, \theta)(0 \leq p, \theta < \infty)$ is the least positive root of the equation

$$\tan\phi \tan(p\theta\phi) = p \tag{36}$$

(thus, $0 < \phi(p, \theta) < \pi/2$).

a) The lower bound of $\alpha_1(\sigma)$ and the upper bound of $\delta_1(\sigma)$ are attained "uniquely" for $\sqrt{c/l}$ with a bang-bang x variation, i.e., for

$$K(L) = \sqrt{\frac{l(L)}{c(L)}} = \begin{cases} K_M & \text{for} \quad L \in (0, Q) \\ K_m & \text{for} \quad L \in (Q, L_T) \end{cases}. \tag{37}$$

Note that, for this case, the plot of the function $C = \sigma(L)$ is the broken line $O\Pi P$ of Fig. 8.

In general, the minimum of $\alpha_n(\sigma)$ and the maximum of $\delta_n(\sigma)$ are reached "uniquely" for the following taper

$$K(L) = \sqrt{\frac{l(L)}{c(L)}} = \begin{cases} K_M & \text{for } L - \dfrac{2kL_T}{2n-1} \in \left(0, \dfrac{Q}{2n-1}\right) \\ K_m & \text{for } L - \dfrac{2kL_T}{2n-1} \in \left(\dfrac{Q}{2n-1}, \dfrac{2q+Q}{2n-1}\right) \\ K_M & \text{for } L - \dfrac{2kL_T}{2n-1} \in \left(\dfrac{2L_T - Q}{2n-1}, \dfrac{2L_T}{2n-1}\right) \end{cases} \tag{38}$$

where $k = 0, 1, 2, \cdots, (n-2)$; and

$$K(L) = \sqrt{\frac{l(L)}{c(L)}} = \begin{cases} K_M & \text{for } L - \dfrac{2(n-1)L_T}{2n-1} \in \left(0, \dfrac{Q}{2n-1}\right) \\ K_m & \text{for } L - \dfrac{2(n-1)L_T}{2n-1} \in \left(\dfrac{Q}{2n-1}, \dfrac{L_T}{2n-1}\right) \end{cases}. \tag{39}$$

The upper bound of $\alpha_1(\sigma)$ and the lower bound of $\delta_1(\sigma)$ are "uniquely" attained for $C = \sigma(L)$ whose plot is the broken line OTP of Fig. 8, i.e., for

$$K(L) = \sqrt{\frac{l(L)}{c(L)}} = \begin{cases} K_m & \text{for} \quad L \in (0, q) \\ K_M & \text{for} \quad L \in (q, L_T). \end{cases} \tag{40}$$

The maximum of $\alpha_n(\sigma)$, and simultaneously the minimum of $\delta_n(\sigma)$, are reached for $\sqrt{l/c}$ defined as in (39) and (40) with K_M and K_m interchanged as well as Q and q.

b) The minimum of $\beta_1(\sigma)$ and maximum of $\gamma_1(\sigma)$ are achieved for the same symmetric taper:

$$K(L) = \sqrt{\frac{l(L)}{c(L)}} = \begin{cases} K_M & \text{for} \quad L \in \left(0, \dfrac{Q}{2}\right) \\ K_m & \text{for} \quad L \in \left(\dfrac{Q}{2}, q + \dfrac{Q}{2}\right) \\ K_M & \text{for} \quad L \in \left(L_T - \dfrac{Q}{2}, L_T\right) \end{cases}. \tag{41}$$

Note that the plot of the function $C = \sigma(L)$ is the broken line $OK\Lambda MP$ in Fig. 8.

In general, if we divide the interval $(0, L_T)$ into n equal parts and in each of them define $K(L)$ analogously to relation (41) using however q/n and Q/n instead of q and Q, we get the "unique" function for which $\beta_n(\sigma)$ reaches its lower bound and $\gamma_n(\sigma)$ its upper bound.

If we interchange Q with q and K_M with K_m in this definition we get the "unique" taper $\in T(L_T, C_T, K_M, K_m)$ for which $\beta_n(\sigma)$ reaches its maximum and $\gamma_n(\sigma)$ its minimum value. Note that for $\beta_1(\sigma)$: maximum and $\gamma_1(\sigma)$: minimum the function $C=\sigma(L)$ is the piecewise linear curve $0\Xi\Lambda NP$ of Fig. 8.

Remark 1: If a_1 is known, using the bounds (31), (32), and the relation

$$\sum_{i=1}^{\infty} 1/\alpha_i^2 = a_1$$

we can find a sharper result for the interval where α_1 must lie, i.e.,

$$\frac{1}{a_1 - \dfrac{0.2337Q^2}{\left[K_M\phi\left(\dfrac{K_M}{K_m}, \dfrac{q}{Q}\right)\right]^2}} \le \alpha_1^2 \le \frac{1}{a_1 - \dfrac{0.2338q^2}{\left[K_M\phi\left(\dfrac{K_m}{K_M}, \dfrac{Q}{q}\right)\right]^2}} \tag{42}$$

where

$$0.2337 \cong \frac{\pi^2}{8} - 1 = \sum_{n=2}^{\infty} \frac{1}{(2n-1)^2}.$$

Similar results can be obtained for the first zero of $B(s)$, $C(s)$, and $D(s)$.

Remark 2: The special case where $K_M=\infty$ corresponds to that in which lumped series inductances are allowed. The bounds on the zeros, α_n, β_n, γ_n, and δ_n can be obtained as before. The case where $K_m=0$ corresponds to that in which lumped shunt capacitances are allowed. A detailed discussion of the bounds on the zeros of both cases is found in [20].

VI. Asymptotic Behavior of the *ABCD* Parameters

Consider a nonuniform lossless transmission line for which the local characteristic impedance $K(z)$, expressed as a function of the "electric" length $z=\int_0^x\sqrt{l(x')c(x')}dx'$, has a continuous second derivative. Using the Liouville transformation [7], [11] we can transform the voltage telegraphists' equation (15) into a differential equation of the form

$$u'' + [\mu^2 - q(z)]u = 0.$$

Using the mathematical results for the eigenfunctions of a differential equation of this form [11] we can deduce the following asymptotic expressions for the *ABCD* parameters on the real-frequency axis; for $\omega \to \infty$:

$$A(j\omega) = \sqrt{\frac{K_1}{K_2}}\cos\Delta\omega + 0\left(\frac{1}{\omega}\right) \tag{43}$$

$$B(j\omega) = \sqrt{K_1K_2}\sin\Delta\omega + 0\left(\frac{1}{\omega}\right) \tag{44}$$

$$C(j\omega) = \frac{1}{\sqrt{K_1K_2}}\sin\Delta\omega + 0\left(\frac{1}{\omega}\right) \tag{45}$$

$$D(j\omega) = \sqrt{\frac{K_2}{K_1}}\cos\Delta\omega + 0\left(\frac{1}{\omega}\right) \tag{46}$$

where

$$K_1 = K(0), \quad K_2 = K(\lambda), \quad \text{and} \quad \Delta = \int_0^\lambda \sqrt{lc}\,dx.$$

Many authors have noted results of this kind before for *LC* or *RC* lines [19]–[22]. If the line contains discontinuities of lumped *L*'s or *C*'s we find the asymptotic behavior of the *ABCD* parameters by dividing the line into sections for which the above result is valid and multiply the corresponding *ABCD* matrices.

Using the frequency transformation $s \to \sqrt{s}$, and referring to some results for *RC* lines [20], [21], we can also make the following statements:

Theorem 2:

For a finite section of a nonuniform lossless transmission line with a piecewise twice differentiable $\sigma'(L)$, which contains n lumped elements, inductances and capacitances (excluding possibly a shunt capacitor at $x=0$ and a series inductor at $x=\lambda$), the asymptotic behavior of $A(s)$, for large s, is

$$A(s) \sim Ks^n e^{\Delta^2 s}, \qquad s \to \infty, \qquad |\arg(s)| \le \frac{\pi}{4} \tag{47}$$

where K is a positive constant and

$$\Delta = \int_0^\lambda \sqrt{l(x)c(x)}\,dx = \int_0^{L_T} \sqrt{\sigma'(L)}\,dL = \int_0^{L_T} K(L)dL. \tag{48}$$

Similarly,

$$B(s) \sim K_1 s^{n_1-1} e^{\Delta^2 s}$$
$$C(s) \sim K_2 s^{n_2+1} e^{\Delta^2 s} \tag{49}$$
$$D(s) \sim K_3 s^{n_3} e^{\Delta^2 s}$$

for $s \to \infty$, $|\arg(s)| \le \pi/4$ where K_1, K_2, K_3 are positive constants and n_1, n_2, n_3 are the number of lumped elements excluding possibly: for n_1, shunt capacitors at both ends; for n_2 series inductors at both ends; and for n_3, a series inductor at $x=0$ and a shunt capacitor at $n=\lambda$.

Corollary 1:

For real $s \to \infty$

$$\log A(s) \sim \log B(s) \sim \log C(s) \sim \log D(s) \sim \Delta s. \quad (50)$$

Theorem 3:

The *ABCD* parameters of an arbitrarily tapered lossless transmission line are entire functions of s of:

1) order $\rho = 1$,

2) type $\sigma = \Delta^2 = \left(\int_0^l \sqrt{l(x)c(x)}\, dx\right)^2 = \left(\int_0^{L_T} K(L)dL\right)^2$,

and

3) genus = 1.

They can be expanded into infinite products as follows:

a) $$A(s) = \prod_{n=1}^{\infty}\left(1 + \frac{s^2}{\alpha_n^2}\right) \quad (51)$$

with $0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots$ and $\alpha_n \sim \frac{\pi n}{\Delta}$ for $n \to \infty$ (52)

b) $$B(s) = L_T s \prod_{n=1}^{\infty}\left(1 + \frac{s^2}{\beta_n^2}\right) \quad (53)$$

with $0 < \beta_1 < \beta_2 < \beta_3 < \cdots$ and $\beta_n \sim \frac{\pi n}{\Delta}$ for $n \to \infty$ (54)

c) $$C(s) = C_T s \prod_{n=1}^{\infty}\left(1 + \frac{s^2}{\gamma_n^2}\right) \quad (55)$$

with $0 < \gamma_1 < \gamma_2 < \gamma_3 < \cdots \gamma_n \sim \frac{\pi n}{\Delta}$ for $n \to \infty$ (56)

d) $$D(s) = \prod_{n=1}^{\infty}\left(1 + \frac{s^2}{\delta_n^2}\right) \quad (57)$$

with $0 < \delta_1 < \delta_2 < \delta_3 < \cdots \delta_n \sim \frac{\pi n}{\Delta}$ for $n \to \infty$. (58)

VII. Computation of the Natural Frequencies

The sums of the even negative powers of the zeros of the *ABCD* parameters can be expressed in terms of the coefficients of the series (1), (2), (3), and (4).

Specifically we have:

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} = a_1, \qquad \sum_{n=1}^{\infty} \frac{1}{\alpha_n^4} = a_1^2 - 2a_2,$$
$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n^6} = a_1^3 - 3a_1a_2 + 3a_3, \quad \text{etc.} \quad (59)$$

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \frac{b_1}{L_T}, \qquad \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} = \frac{b_1^2 - 2b_2}{L_T},$$
$$\sum_{n=1}^{\infty} \frac{1}{\beta_n^3} = \frac{b_1^3 - 3b_1b_2 + 3b_3}{L_T}, \quad \text{etc.} \quad (60)$$

Similar relations hold for the zeros of $D(s)$ and $C(s)$.

An approximate method for the computation of the zeros of the *ABCD* parameters can be based on (59) and (60). Taking under consideration that, beginning with a sufficiently large subscript, all zeros may be regarded as known, i.e., as equal to their known asymptotic values, then an algebraic system of equations is obtained from (59), (60), etc., for the first few zeros.

To be specific, suppose that we want to compute approximately the first few zeros of $B(s)$. The easiest asymptotic estimate of β_n for large n, is $\beta_n \sim \pi n/\Delta$.

Suppose that this asymptotic formula gives a "good" approximate value of β_n beginning with $n = 4$. Then we will have for the first three zeros of $B(s)$ the following algebraic system.

$$\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2} + \frac{1}{\beta_3^2} = \frac{b_1}{L_T} - \sum_{n=4}^{\infty} \frac{1}{\beta_n^2} \cong \frac{b_1}{L_T} - \frac{\Delta^2}{\pi^2} \sum_{n=4}^{\infty} \frac{1}{n^2} = \frac{b_1}{b_0} - \left(\frac{\Delta}{\pi}\right)^2 \zeta(2, 4) \equiv \Lambda$$

$$\frac{1}{\beta_1^4} + \frac{1}{\beta_2^4} + \frac{1}{\beta_3^4} = \frac{b_1^2 - 2b_2}{L_T} - \sum_{n=4}^{\infty} \frac{1}{\beta_n^4} \cong \frac{b_1^2 - 2b_2}{L_T} - \left(\frac{\Delta}{\pi}\right)^2 \zeta(4, 4) \equiv M \quad (61)$$

$$\frac{1}{\beta_1^6} + \frac{1}{\beta_2^6} + \frac{1}{\beta_3^6} = \frac{b_1^3 - 3b_1b_2 + 3b_3}{L_T} - \sum_{n=4}^{\infty} \frac{1}{\beta_n^3} = \frac{b_1^3 - 3b_1b_2 + 3b_3}{L_T} - \left(\frac{\Delta}{\pi}\right)^3 \zeta(6, 4) \equiv N.$$

Where $\zeta(s, \nu)$ is the modified Riemann's Zeta function defined as follows

$$\zeta(s, \nu) = \sum_{n=0}^{\infty} \frac{1}{(n + \nu)^s} \tag{62}$$

for ν, positive integer

$$\zeta(s, \nu) = \sum_{n=\nu}^{\infty} \frac{1}{n^s} \cdot \tag{63}$$

The values of $1/\beta_1^2$, $1/\beta_2^2$, and $1/\beta_3^2$ which satisfy the algebraic system (61) are the roots of the third order equation

$$x^3 - \Lambda x^2 + \frac{\Lambda^2 - M}{2} x + \frac{\Lambda^3 + 2N - 3\Lambda N}{6} = 0.$$

VIII. Conclusion

We have presented a method of analysis of the general nonuniform transmission line. In contrast with most of the known methods, the present method does not require the solution of any nonlinear differential equation. Secondly, the present formulation leads naturally to the relationship between the natural frequencies of the line and the eigenvalues of a Sturm-Liouville problem. Finally, the analytic properties of the network functions of nonuniform line are important in the study of the realizability and the synthesis of nonuniform lines.

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